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New Exact Solutions of Heat Conduction in Metals

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We consider the way in which a solution to a class of nonlinear partial differential equations $S(u)u_t = (K(u)u_x)_x$ approaches the similarity form. The problem we solve is chosen for two main reasons: first the equation above is of widespread use in modeling physical situations and second it provides a tractable but significant example of a free boundary problem.

KEY WORDS: Heat conduction, invariants of the symmetries; similarity reduction.

1. INTRODUCTION

The problem of heat conduction in isotropic solids and the processes of melting and evaporation of metals in the case that their surface is exposed to a powerful flux of energy are described by a nonlinear equation.[1–4]

$$S(u)u_t = (K(u)u_x)_x, \qquad t \geqslant 0 \quad \text{and} \quad x \geqslant 0 \tag{1.1}$$

where $K(u)$ is the thermal conductivity, $S(u)$ is the specific heat, and $u(x, t)$ is the temperature field to be found. A transformation which could in certain cases linearize the nonlinear heat equation was first introduced by Storm[1] for a constant heat flux. Using the Backlund transformation, which leads to the theory of homology, Munier *et. al*[3] gave an example based on heat conduction in metals for the special case, where the product of the thermal coefficients SK is constant, and Eq. (1.1) in this case can be transformed into the linear heat equation.

Cherniha *et. al*[4] devised a special transformation $s = (x - vt)$, which reduced Eq. (1.1) to nonlinear ordinary differential equation for some functions $S(u)$ and $K(u)$.

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Physically interesting problems usually have some symmetries. Using these symmetries; we can simplify the equation to a certain extent. There is a systematic way to do this, viz., using Lie group analysis to find all the invariants of the symmetries, and constructing solutions with these invariants. In this way the equations will be simplified and special solutions can be obtained. The similarity transformation method based on Lie group analysis has many applications when dealing with differential equations and related physical problems.[5-9] Especially in the nonlinear case, it can sometimes help us in finding physically meaningful exact solutions.

The equation (1.1) we are going to consider is strongly nonlinear, and it is desirable and interesting to find all analytic solutions or reduce it to ordinary differential equations that may be solved numerically. One purpose of this study is to find all transformations which will reduce the heat conduction equation (1.1), with a general form of the thermal coefficients $S(u)$ and $K(u)$. The central motivation of this article is to apply the Lie similarity method, which does not seem to have been applied systematically to the nonlinear equation (1.1).

The present article is organized as follows: In Section 2 an exhaustive symmetry analysis is carried out and this enables us to classify the thermal coefficients according to the symmetry properties of Eq. (1.1). Lie group generators for every case are found, the basic fields of an optimal system are given, and the corresponding subclasses of reduced ordinary differential equations are presented in tabular form. In Section 3 we construct exact classes of similarity solutions.

2. SYMMETRY TRANSFORMATION AND SIMILARITY REDUCTIONS

Consider the one-parameter (ε) Lie group of infinitesimal transformations in x, t and u given by

$$\underline{x} = x + \varepsilon X(x, t, u,) + O(\varepsilon^2) \tag{2.1a}$$

$$\underline{t} = t + \varepsilon T(x, t, u) + O(\varepsilon^2) \tag{2.1b}$$

$$\underline{u} = u + \varepsilon U(x, t, u) + O(\varepsilon^2) \tag{2.1c}$$

Then the first and second extensions, which refer to how the first and second partial derivatives transform, can be determined by using Eqs. (2.1), for example

$$\underline{u}_{\underline{x}} = \frac{\partial(u(x, t) + \varepsilon U(x, t, u))}{\partial_{\underline{x}}} + O(\varepsilon^2)$$
$$= u_x + \varepsilon(U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t) + O(\varepsilon^2)$$

which we write as

$$\underline{u}_{\underline{x}} = u_x + \varepsilon U^x + O(\varepsilon^2) \tag{2.2a}$$

where U^x denotes the infinitesimal transformation of $\underline{u}_{\underline{x}}$. In a similar way, the infinitesimal transformations U^t, U^{xx} of the partial derivatives $\underline{u}_{\underline{t}}$ and $\underline{u}_{\underline{xx}}$ can be obtained[5] and we have

$$\underline{u}_{\underline{t}} = u_t + \varepsilon U^t + O(\varepsilon^2) \tag{2.2b}$$

$$\underline{u}_{\underline{xx}} = u_{xx} + \varepsilon U^{xx} + O(\varepsilon^2) \tag{2.2c}$$

where

$$U^t = U_t + (U_u - T_t) u_t - X_t u_x - T_u u_t^2 - X_u u_x u_t) \tag{2.2d}$$

$$\begin{aligned} U^{xx} = {} & U_{xx} + (2U_{xu} - X_{xx}) u_x - T_{xx} u_t + (U_{uu} - 2X_{xu}) u_x^2 \\ & - 2T_{xu} u_x u_t - X_{uu} u_x^3 - T_{uu} u_x^2 u_t + (U_u - 2X_x) u_{xx} \\ & - 2T_x u_{xt} - 3X_u u_{xx} u_x - T_u u_{xx} u_t - 2T_u u_{xt} u_x \end{aligned} \tag{2.2e}$$

The nonlinear heat equation (1.1) is invariant under this transformation if

$$S(\underline{u}) \underline{u}_{\underline{t}} = (K(\underline{u}) \underline{u}_{\underline{x}})_{\underline{x}} \tag{2.3}$$

Making use of Eqs. (2.1) and Eqs. (2.2), to first order in ε, Eq. (2.3) becomes

$$SU^t + Uu_t\, ds/du = KU^{xx} + 2U^x u_x\, dK/du + Uu_{xx}^2\, dK/du + Uu_x^2 d^2K/du^2 \tag{2.4}$$

Conditions on the infinitesimals X, T and U are determined by equating coefficients of like monomials in U_x and u_t and higher derivatives. This leads to a set of partial differential equations that must then be solved. After some simplification, they given by,

$$T_x = T_u = 0, \qquad X_u = 0 \tag{2.4a}$$

$$U_{uu} + d(\ln K)/du\ U_u = d^2(\ln K)/du^2\ U = 0 \tag{2.4b}$$

$$U = (2X_x - T_t)/[d(\ln(k/S))/du] \tag{2.4c}$$

$$U_t = (K/S)\ U_{xx} \tag{2.4d}$$

$$2(dK/du)\ U_x + SX_t + K(2U_{xu} - X_{xx}) = 0 \tag{2.4e}$$

Analysis of this system of equations leads to the explicit from of the functions X, T and U. For example, substituting Eq. (2.4c) into both of

Eqs. (2.4e) and Eqs. (2.4.d) and equating different powers of u to zero gives

$$X_{xx}=0, \qquad X_t=0, \qquad T_{tt}=0$$

which imply that

$$X=X(x)=C_1x+C_2 \tag{2.5a}$$

$$T=T(t)=C_3t+C_4 \tag{2.5b}$$

$$U=U(u)=(2C_1-C_3)/d[\ln(K/S)]/du \tag{2.5c}$$

where C_i's are arbitrary constants . Substitution of Eq. (2.5c) into Eq. (2.4b) gives

$$Y'''/Y'-2(Y''/Y')^2+(Y''/Y')(K'/K)-(K'/K)'=0 \tag{2.6}$$

where $Y=\ln(K/S)$ and primes denote differentiation with respect to u. Four cases arise depending on whether or not $K(u)$ and $S(u)$ satisfy equation (2.6); these are:

$$\begin{array}{lll} \text{Case 1:} & S(u)=a+bu, & K(u)=c \\ \text{Case 2:} & S(u)=\exp(au), & K(u)=b \\ \text{Case 3:} & S(u)=u^r & K(u)=u^q \\ \text{Case 4:} & S(u)=\exp(au), & K(u)=\exp(bu) \end{array} \tag{2.7}$$

where a, b, c, r, and q are arbitrary constants.

The knowledge of the infinitesimal elements X, T and U given in Eq. (2.5) enables us to construct the differential operator of the form

$$B=X\partial/\partial x+T\partial/\partial t+U\partial/\partial u$$

which depends on the number of the group constants C_1, C_2, C_3, and C_4. The four generating operators B_i can be constructed by taking one of the group constants equal to 1 and the remaining three constants equal to zero. One obtains

$$\begin{aligned} B_1&=\partial_x \\ B_2&=\partial_t \\ B_3&=x\partial_x+(2/Y')\partial_u \\ B_4&=t\partial_t-(1/Y')\partial_u \end{aligned} \tag{2.8}$$

Table I. Symmetries of the One-Dimensional Nonlinear Heat Conduction Equation[a]

$S(u)$	$K(u)$	$B_3, B_4 = X\partial_x + T\partial_t + U\partial_u$
$a+bu$	c	$B_3 = x\partial_x - 2(u+a/b)\,\partial_u$
		$B_4 = t\partial_t + (u+a/b)\,\partial_u$
e^{au}	b	$B_3 = x\partial_x + (2/a)\,\partial_u$
		$B_4 = t\partial_t + (1/a)\,\partial_u$
u^r	u^q	$B_3 = x\partial_x + (2u/q - r)\,\partial_u$
		$B_4 = t\partial_t - (u/q - r)\,\partial_u$
e^{au}	e^{bu}	$B_3 = x\partial_x + [2/(b-a)]\,\partial_u$
		$B_4 = t\partial_t - [1/(b-a)]\,\partial_u$

[a] $a, b, c, r, q \in R$.

These four linearly independent operators determine the symmetries under which Eq. (1.1) is invariant . The two symmetries B_1 and B_2 apply for all forms of $S(u)$ and $K(u)$, and are omitted from Table I, where B_3 and B_4 are shown for the different cases in (2.7).

The main use of symmetries is to obtain a reduction of variables in Eq. (1.1), which can be obtained by solving the following characteristic equation:

$$dx/X = dt/T = du/U$$

The general solution of these equations will involve two arbitrary constants, one of which takes the role of similarity variable $s = s(x, t)$ and the other, say $F(s)$, plays the role of dependent variable, usually called the similarity function. By substituting the similarity forms in the partial differential equation (1.1), it will be reduced to an ordinary differential equation in $F(s)$. Solutions $F(s)$ lead by back substitution to so-called

Table II. The Optimal System for Heat Equation (1.1)

$S(u)$.	$K(u)$	Optimal system
$a+bu$	c	$B_3,\ B_4,\ B_1+B_2,\ B_3+B_2,\ B_1+B_4$
e^{au}	b	$B_4,\ B_3,\ B_1+B_2,\ B_1+B_4,\ B_2+B_3$
u^r	u^q	$B_2,\ B_3,\ B_4,\ B_1+B_2,\ B_1+B_4,\ B_4+B_3,\ B_2+B_3$
e^{au}	e^{bu}	$B_2,\ B_3,\ B_4,\ B_1+B_2,\ B_3+B_4,\ B_2+B_3,\ B_1+B_4$

Table III. The Optimal System for Heat Equation (1.1)[a]

	$s(x,t)$	$u(x,t)$	Reduced equations
I. $(a+bu)u_t = cu_{xx}$			
B_3	t	$F/2x^2 - a/b$	$F' = 12c/b$
B_4	x	$tF - a/b$	$F'' = (b/c)F^2$
$B_1 + B_2$	$x - t$	F	$cF'' + aF' + bFF' = 0$
$B_3 + B_2$	e^t/x	$F/x^2 - a/b$	$s^2F'' + 6sF' - (bs/c)\,FF' + 6F = 0$*
$B_1 + B_4$	$e^x t^{-1}$	$tF - a/b$	$s^2F'' + sF' + (b/c)\,sFF' - (b/c)\,F^2 = 0$*
II. $\exp(au)\, u_t = bu_{xx}$			
B_3	t	$\ln(x^{-2/a}F)$	$F^{a-1}F' = 2b/a$
B_4	x	$\ln(t^{1/a}F)$	$FF'' - F'^2 = F^{a+2}/ab$
$B_1 + B_2$	$x - t$	F	$bF'' + e^{aF}F' = 0$
$B_1 + B_4$	e^x/t	$\ln(t^{1/a}F)$	$bs^2FF'' + bsFF' - bs^2F'^2 + sF^{a+1}F'$ $- F^{a+2}/a = 0$*
$B_2 + B_3$	$e^t x^{-1}$	$\ln(x^{-2/a}F)$	$bs^2FF'' - bs^2F'^2 + 2bsFF' + (2ba)\,F^2$ $- sF^{a+1}F' = 0$*
III. $u^r u_t = (u^q u_x)_x$			
B_3	t	$x^{2/(q-r)}F$	$F' = \{(2q + 2r + 4)/(q - r^2)\}\, F^{q-r+1}$
B_2	x	F	$FF'' + qF'^2 = 0$
B_4	x	$t^{-1/(q-r)F}$	$FF'' + qF'^2 + F^{r-q+2}/(q-r) = 0$
$B_1 + B_2$	$x - t$	F	$FF'' + qF'^2 + F^{r-q+1}F' = 0$
$B_1 + B_4$	$e^x t^{-1}$	$e^{x/(r-q)}F$	$s^2FF'' + qs^2F'^2 + s^2F'F^{r-q+1}$ $+ \{(q + r - 2)/(r - q)\}\, sFF' + \{(q+1)/(q-r)^2\}\, F^2 = 0$*
$B_3 + B_4$	xt^{-1}	$x^{1/(q-r)}F$	$s^2FF'' + qs^2F'^2 + s^2F'F^{r-q+1}$ $+ \{(2 + 2q)/(q - r)\}\, sFF' + \{(1 + r)/(q - r)^2\}\, F^2 = 0$*
$B_2 + B_3$	xe^{-t}	$e^{2t/(q-r)}F$	$FF'' + qF'^2 + sF^{r-q+1}F'[2/(q-r)]\, F^{2-q+r} = 0$*
IV. $\exp(au)\, u_t = [\exp(bu)\, u_x]_x$			
B_3	t	$\ln(x^{2/(b-a)}F)$	$F^{a-b-1}F' = (2a + 2b)/(a-b)^2$
B_4	x	$\ln(t^{1/(a-b)}F)$	$FF'' + (b-1)\,F'^2 - F^{a-b+2}/(a-b) = 0$
B_2	x	F	$F'' + bF'^2 = 0$
$B_1 + B_2$	$t - x$	F	$e^{(b-a)F}(F'' + bF'^2) = F'$
$B_3 + B_4$	t/x	$\ln(x^{-1/(a-b)}F)$	$s^2FF'' + \{2as/(a-b)\}\, FF'$ $+ (b-1)\, s^2F'^2 + \{a/(b-a)^2\}\, F^2 - F^{a-b+1}F' = 0$*
$B_2 + B_3$	xe^{-t}	$\ln(x^{-2/(a-b)}F)$	$s^2FF'' + (b-1)\, s^2F'^2 + sF^{a-b+1}F'$ $- \{4bs/(b-a)\}\, FF' + \{2(a+b)/(b-a)^2\}\, F^2 = 0$*
$B_1 + B_4$	te^{-x}	$\ln(t^{1/(a-b)}F)$	$s^2FF'' + sFF' - (b+1)\, s^2F'^2 - sF^{a-b+1}F'$ $- F^{a-b+2}/(a-b) = 0$*

[a] Here $F' = dF/ds$. Asterisk indicates nonintegrable ODE.

similarity solutions $u(x, t)$ of Eq. (1.1). Reductions of Eq. (1.1) may be obtained from any symmetry which is an arbitrary linear combination

$$d_1 B_1 + d_2 B_2 + d_3 B_3 + d_4 B_4, \qquad d_i \in R$$

Since there are almost always an infinite number of such combinations it is usually not feasible to list all possible similarity solutions. A systematic procedure of classifying these solutions is based on the property that the transformations of the symmetry group will transform solutions of the differential equation into solutions (see, e.g., ref. 9). Therefore, it is sufficient to consider only linear combinations which lead to solutions that are inequivalent with respect to symmetry transformations, this set of solutions is called an optimal system. The optimal system is determined for each of the cases listed in Table I to ensure that a minimal complete set of reductions is obtained from the symmetries of the governing equation. Table II lists the optimal system from each of the entries from Table I.

Table III shows the reduced ordinary differential equation and relates symmetry invariants for each of the optimal systems in Table II, together with the corresponding similarity variables s and the similarity forms connecting $F(s)$ and $u(x, t)$. The reduced ODEs listed in Table III may or may not be solvable in closed from.

3. EXPLICIT SIMILARITY SOLUTIONS OF Eq. (1.1)

The resulting ordinary differential equations in Table III are of second order, except for the case where the symmetry B_3 with $s = t$ leads to a first order ODE, and by separating the dependent and independent variables they can be reduced to quadrature. Thus the explicit general solutions are simple. The ODEs of second order resulting from the reductions are nonlinear; some of them belong to the class of integrable and exactly solvable evolution equations, and the others are not integrable but may be solved by numerical methods. The nonintegrable ODEs in Table III are marked by asterisks. In the following we will focus our attention on the analytic solution of the integrable nonlinear ODEs listed in Table III in the cases mentioned earlier.

Case 1. If $S(u) = a + bu$, $K(u) = c$; $c, b \neq 0$. The nonlinear diffusion equation (1.1) in this form occurs in the problem of the thermal expulsion of fluid from a long, slender, heated tube,[10] and the quantity u represents the flow velocity induced in the fluid by the heating of the tube wall.

The second equation has the solution

$$F(s) = [(b/6c)^{-1/2} s + c]^{-2}$$

C is a constant. Then Eq. (1.1) has a solution of the form

$$u(x, t) = tF(x) - a/b \tag{3.1}$$

Another type of solution can be obtained by symmetry $B_1 + B_2$, where the corresponding reduced equation has the solution

$$F(s) = (Ae^{Gs} - D)/1 - C_1 e^{Gs})$$

where A, G and D given by

$$bA = C_1[a - (a^2 + 2bC_2)^{-1/2}]$$

$$cG = (a^2 + 2bC_2)^{-1/2}$$

$$bD = a + (a^2 + 2bC_2)^{-1/2}$$

and C_1 and C_2 are constants of integration. From the similarity representation in Table III, Eq. (1.1) has the solution,

$$u(x, t) = F(x - t) = (Ae^{G(x-t)} - D)/(1 - C_1 e^{G(x-t)}) \tag{3.2}$$

This result is very close to the result in ref. 4 with appropriate boundary values.

Case 2. $S(u) = e^{au}$ and $K(u) = b$; $a, b \neq 0$ The second ODE is nonlinear of second order, exact solutions can be found only for special values of the parameters: $a = -2$, 1 and b arbitrary.

(i) For $a = -2$, it has the solution

$$C_1 F(s) = (2b)^{-1/2} \cos(C_1 s + C_2), \qquad b > 0$$

$$C_1 F(s) = (2b)^{-1/2} \cosh(C_1 s + C_2), \qquad b < 0$$

then Eq. (1.1) has the solution

$$u(x, t) = \ln[t^{-1/2} F(x)] \tag{3.3}$$

(ii) For $a=1$, it has the solution

$$2F(s)=bC_1(\tanh^2 s_1-1), \qquad C_1>0$$
$$2F(s)=-bC_1(\tan^2 s_2+1), \qquad C_1<0$$

where $s_1=C_2-\frac{1}{2}s\sqrt{C_1}$ *and* $s_2=C_2+\frac{1}{2}s\sqrt{-C_1}$, with C_1 and C_2 constants. Then Eq. 1 has the solution

$$u(x, t)=\ln[tF(x)] \tag{3.4}$$

The third reduced equation in Table III for this case has the solution $F(s)=\ln(s/b-C_1)^{-1/a}$, $C_1=$ constant and the second integration constant is zero . Then Eq. 1 has the solution

$$u(x, t)=\ln[(x-t)/b-C_1]^{-1/a} \tag{3.5}$$

The range of the variables is determined by requiring the expression in the r.h.s. to be positive . These results may be compared to the results in ref. 4.

Case 3. $S(u)=u^r$ and $K(u)=u^q$. A different variety of solutions can be determined by solving the reduced ODEs in Table III as follows: The second equation has the solution

$$\begin{aligned} F(s)&=\exp(C_1+C_2 s) \qquad \text{for} \quad q=-1 \\ F(s)&=(C_1+C_2 s)^{1/q+1} \qquad \text{for} \quad q\neq -1 \end{aligned} \tag{3.6}$$

Then Eq. (1.1) has the stationary solution $u=F(x)$, with C_1, C_2 constants. The third equation is integrable for two cases:

(i) When $r-q=1$, the first integral gives

$$F'^2=(C_1F^{-2q}-4qF^3)/[-2q(3+2q)], \qquad C_1=\text{const} \tag{3.7}$$

which can be integrated once again for certain values of q.

(ii) when $r-q=2$, the first integral gives

$$F'^2=C_1F^{-2q}+F^4/(2q+4) \qquad C_1=\text{const} \tag{3.8}$$

which can be integrated once again for certain values of q. Once we have the solution of the reduced equation $F(s)$, the general solution of Eq. (1,1) is given by

$$u(x, t)=t^{-1/q-r}F(x)$$

The fourth ODE, has a first integral given by

$$F' = C_1 F^{-q}, \; -F^{-q+r+1}/(r+1) \qquad r \neq -1 \tag{3.9}$$

which is separable and can be integrated once again for some values of r and q.

It is worth noting that Eq. (1.1) in this case, with $r = 1$, describes diffusion of a species. It has been pointed out by Dresner[10] that this equation arises in the problem of the current distribution in superconductors undergoing a current ramp.

Case 4. $S(u) = e^{au}$ and $K(u) = e^{bu}$, $a \neq b$. The third ODE has the solution

$$F = \ln(C_1 s + C_2)^{1/b}$$

where C_1 and C_2 are constants. The stationary solution of Eq. (1.1) is given by

$$u(x, t) = \ln(C_1 x + C_2)^{1/b} \tag{3.10}$$

The fourth ODE has a first integral of the form

$$F' = e^{-bF}(C_1 + e^{aF}/a) \tag{3.11}$$

where C_1 is constant of integration. Equation (3.11) is a separable ODE and can be integrated once again for some values of a and b.

The second ODE is integrable for some values of a and b as follows:

1. For $b = 0, a = -2$, it has the solution

$$F(s) = \cos(C_1 s + C_2)/\sqrt{2}\, C_1$$

where C_1 and C_2 are constants. Then Eq. (1.1) has the solution

$$u = \ln[t^{-1/2} F(x)] \tag{3.12}$$

2. For $b = 1, a = -1$ it has the first integral

$$F'^2 = \ln F^{-1} + C_1 \tag{3.13}$$

3. For $b = 2, a = 0$, its solution is

$$F^2 = C_1 + C_2 s - s^2/2 \tag{3.14}$$

where C_1 and C_2 are constants.

4. For $b=0$, $a=-1$, it has first integral of the form

$$F'^2 = 2F + C_1 F^2 \tag{3.15}$$

5. For $b=0$, $a=1$, its first integral is

$$F'^2 = 2F^3 + C_1 F^2 \tag{3.16}$$

where C_1 is a constant.

Equations (3.13) (3.15) and (3.16) are separable and may be integrated again to get the explicit solutions $F(s)$. Once we know $F(s)$, the similarity solution $u(x, t)$ of Eq.(1.1) can be obtained by using the similarity representation in Table III.

It is worth noting that the solution of Eq. (1.1) in this case has already been given by Storm[1] for the constant heat flux $S=K=$const. and by Munier *et. al.*[3] for the case $S(u)=e^{-mu}$ and $K(u)=e^{mu}$. Here we have determined some new exact solutions for the general form of the thermal coefficient S and K.

4. CONCLUSIONS

The results derived in this paper all stem from the enlargement of the generalization of Lie's classical method using symmetry groups. The detailed analysis of similarity solutions associated with classical groups of the heat conduction equation (1.1) indicates the following result: invariance principles, through the Lie similarity method, lead to a classification of the thermal coefficients and enables us to obtain a great variety of solutions to a nonlinear problem. In addition, they asymptotically describe the evolution of the problem with quite general initial or boundary conditions. As a final comment, we notice that an application of the Lie similarity method puts us inlet a position to construct the most general classes of similarity solutions for Eq. (1.1) and represents one of the most powerful analytical techniques to solve nonlinear differential equations.

A procedure to obtain new exact solutions by the Lie similarity method has also been successfully applied to invariant partial differential equations under potential symmetries.[12, 13]

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